THE
MATHEMATICAL GAZETTE.

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THE NORMAL LAW OF ERROR.

1. The investigations of Professor Karl Pearson and his coadjutors on the applications of the theory of errors to statistics have greatly increased the importance and extent of the subject.

It is hoped that the following version of one of the methods of arriving at the so-called normal law of error may be of interest from an elementary point of view: affording, as it does, a simple application of the binomial theorem and some interesting graphical work.

Let each single observation of a quantity, for instance the measurement of a length, be subject to $2n$ errors, each of amount z and each equally likely, in any one case, to be positive or negative.

If $n+r$ errors are positive and $n-r$ are negative, the total, or resultant error will be $2rz$. Now the number of permutations taken all together of $(n+r)$ + signs and $(n-r)$ − signs is $\frac{2n!}{(n+r)!(n-r)!}$, and each permutation represents a mode of occurrence of the set of errors, the chance of that particular permutation occurring being $\frac{1}{2^{2n}}$.

Hence the chance of a total error $2rz = \frac{2n}{n+r!\,n-r!}\,\frac{1}{2^{2n}}$. In other words, the chances of the various total errors

$$2nz,\ 2\overline{n-1}z\ \ldots,\ 2rz\ \ldots,\ -2nz,$$

are given by the successive terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{2n}$.

2. Let us examine the mean value, or expectation, say σ^2, of the square* of the total error.

Multiplying the chance of each total error by the square of its amount, and summing the results we have

$$\sigma^2=\sum_{r=-n}^{r=n}\frac{2n!}{(n+r)!(n-r)!}\frac{1}{2^{2n}}4r^2z^2$$

$$=\sum_{r=-n}^{r=n}\frac{2n!}{(n+r)!(n-r)!}\frac{1}{2^{2n}}[n^2-(n^2-r^2)]4z^2$$

$$=\frac{4n^2z^2}{2^{2n}}\sum_{r=-n}^{r=n}\frac{2n!}{(n+r)!(n-r)!}-\sum_{r=-n}^{r=n}\frac{2n!}{(n+r-1)!(n-r-1)!}\cdot\frac{1}{2^{2n}}\cdot 4z^2$$

$$=\frac{4n^2z^2}{2^{2n}}(1+1)^{2n}-\frac{2n\,.\,2n-1}{2^{2n}}4z^2(1+1)^{2n-2}$$

$$=4n^2z^2-2n\,.\,(2n-1)z^2$$

$$=2nz^2.$$

If then n and z are connected by the equation

$$2nz^2=\text{constant}=\sigma^2,$$

the "standard deviation" σ will be the same for all values of n.

3. It is interesting to represent the chances of errors of given amount graphically, taking different values of n and z, and keeping σ constant.

Thus we may take $\quad \sigma=4$ inches,

and the cases $\quad n=8,\ 32,\ 18,$

$$z=1,\ \tfrac{1}{2},\ \tfrac{2}{3}.$$

Taking the first case the maximum ordinate is proportional to $\frac{16!}{8!8!}$ and taking 10 inches to represent this, we have, if $r=0$, say $y_0=10$ inches.

If $r=1$, y_1 is proportional to $\frac{16!}{7!9!}$ and therefore $y_1=\frac{8}{9}y_0$ $=8{\cdot}89$, y_1 corresponding to a total error x_1 of 2 inches.

If $r=2$, y_2 is proportional to $\frac{16!}{6!10!}$ and therefore $y_2=\frac{7}{10}y_1$ $=6{\cdot}22$ and so on. Using a slide rate the sucessive ordinates are thus rapidly worked out, and the total error x, and the corresponding chance, y, plotted.

Taking in the same way one of the other cases, we find that a fair curve through the second set of points passes very close to the first set also. In other words, provided σ is constant and n

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* It is natural to ask why the *square* should be taken. Prof. Pearson, *Grammar of Science*, p. 386, cites the analogy of the radius of gyration k: k^2 being the mean square of the distance. Of course the simple mean value or expectation of the total error is zero, for positive and negative errors of equal amounts are equally likely.

is fairly large, the chance of getting an error of a given amount at a single trial is, on the above hypothesis, almost independent of the value of n.

4. Let us now prepare to suppose n to become infinite and z infinitely small in such a manner that $2nz^2$ retains the constant value σ, and r to become infinite in such a manner that $2rz=x$.

Let y denote the relative * chance of a total error $x=2rz$,
and $y+\Delta y$ " " " $x+\Delta x=(2r+2)z$.

Thus as before

$$y=\frac{2n!}{(n+r)!(n-r)!}\,\frac{1}{2^{2n}},$$

$$y+\Delta y=\frac{2n!}{(n+r+1)!(n-r-1)!}\,\frac{1}{2^{2n}},$$

whence, by division,

$$1+\frac{\Delta y}{y}=\frac{n-r}{n+r+1} \quad \text{or} \quad \frac{\Delta y}{y}=\frac{-2r-1}{n+r+1}=\frac{-x-z}{nz+\frac{1}{2}x+z},$$

or, since $\Delta x=2z$,

$$\frac{1}{y}\,\frac{\Delta y}{\Delta x}=\frac{-x-z}{2nz^2+xz+2z^2}=\frac{-x-z}{\sigma^2+xz+2z^2}.$$

Now proceed to the limit as above indicated and we have

$$\frac{1}{y}\,\frac{dy}{dx}=-\frac{x}{\sigma^2};$$

or, integrating, $\log y=-\dfrac{x^2}{2\sigma^2}+C,$

or $y=y_0 e^{-\frac{x^2}{2\sigma^2}}.$

Taking $y_0=10$ inches,

$\sigma=4$ inches,

and plotting the curve, we find once more that it passes very closely to the points already found by giving n the finite value 32. If we adjust the value of y_0 so that the whole area of the curve is unity, which should be so, for the area represents the chance of obtaining *some* error between + and − infinity (zero included), we find

$$y_0=\frac{1}{\sigma\cdot\sqrt{2\pi}}.$$

It is now possible to plot, in their proper relative proportions, the curves corresponding to *different* values of σ.

* Of course the *absolute* chance of an error being *exactly* x is zero. The chance that the error lies between x and $x+\Delta x$ is $y\Delta x$ nearly.

Abundant illustrations of the application to statistics of the properties of the Normal Curve of Error

$$y=\frac{1}{\sigma\cdot\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

may be found in Professor Pearson's *Grammar of Science*, and the series of papers in the *Phil. Trans.* already alluded to. Mr. W. F. Sheppard proves many of the properties by elementary methods in a paper in the *Philosophical Transaction*, vol. 192 A, p. 101.

C. S. Jackson.

THE POWER SERIES FOR $\sin x$, $\cos x$.

The hopeless complexity, from the elementary didactic point of view, of the hitherto accepted accurate trigonometrical proofs of the power series for $\sin x$, $\cos x$ has been so fully insisted upon in these pages (this volume, pp. 85, 86), that a more elementary method seems urgently called for. The series in question follow at once from the two theorems

$$\sin x \text{ lies between } S_n,\ S_{n+1}, \qquad (1)$$

$$\cos x \text{ lies between } C_n,\ C_{n+1}, \qquad (2)$$

where

$$S_n=\frac{x}{\underline{|1}}-\frac{x^3}{\underline{|3}}+\ldots+(-1)^n\frac{x^{2n+1}}{\underline{|2n+1}},$$

$$C_n=1-\frac{x^2}{\underline{|2}}+\ldots+(-1)^n\frac{x^{2n}}{\underline{|2n}}.$$

Now it may be shown by elementary methods that (1) follows from (2) for all real values of x. It may also be similarly shown that if (1) be granted, then $\cos x$ lies between C_{n+1}, C_{n+2} provided $-2\pi \leqslant x \leqslant +2\pi$. Since then it is known that (2) is true when $n=0$ for all real values of x, we at once obtain an inductive proof of (1), (2) for $-2\pi\leqslant x\leqslant +2\pi$.

To show that (1) follows from (2) let there be m angles $\alpha, \beta, \gamma, \ldots$, then

$$\cos\alpha\cos\beta\cos\gamma\ldots=\frac{1}{2^m}\Sigma\cos(\pm\alpha\pm\beta\pm\gamma\pm\ldots).$$

Hence, according as $\cos x \gtreqless a_0+\Sigma a_r x^{2r}$,

we have $\cos\alpha\cos\beta\cos\gamma\ldots\gtreqless a_0+\Sigma a_r T_r$,

where $T_r=\frac{1}{2^m}\Sigma(\pm\alpha\pm\beta\pm\gamma\pm\ldots)^{2r}.$

Now taking $\alpha, \beta, \gamma, \ldots$ to be $\frac{x}{2}, \frac{x}{2^2}, \ldots, \frac{x}{2^m}$, we have

$$T_r=\frac{2x^{2r}}{2^{m(2r+1)}}t_r,$$

where t_r is the sum of the $2r^{\text{th}}$ powers of the odd numbers from 1 to 2^m-1, so that

$$t_r=\frac{1}{2r+1}\frac{2^{m(2r+1)}}{2}(1+\epsilon),$$

where ϵ can be made as small as we please by taking m large enough. Thus we have

$$T_r=\frac{x^{2r}}{2r+1}(1+\epsilon).$$

Hence, since $\quad \cos\frac{x}{2}\cos\frac{x}{2^2}\dots\cos\frac{x}{2^m}=\dfrac{\sin x}{2^m\sin\frac{x}{2^m}}$

it follows that $\quad \dfrac{\sin x}{2^m\sin\frac{x}{2^m}}\gtreqless a_0+\Sigma\dfrac{a_r}{2r+1}(1+\epsilon)x^{2r},$

according as $\quad \cos x\gtreqless a_0+\Sigma a_r x^{2r}.$

Now taking $\quad a_r=\dfrac{(-1)^r}{\underline{|2r}}$, $r=0, 1, 2, \dots n$

and making m infinite, we see that (1) follows from (2).

To prove the other result which has been stated we require the elementary algebraic inequality that if u lies between U_n, U_{n+1} and v lies between V_n, V_{n+1}, where

$$U_n=u_0-u_1+u_2-\dots+(-1)^n u_n,$$
$$V_n=v_0-v_1+v_2-\dots+(-1)^n v_n,$$

then uv lies between W_n, W_{n+1}, where

$$W_n=w_0-w_1+w_2-\dots+(-1)^n w_n,$$
$$w_r=u_0v_r+u_1v_{r-1}+\dots+u_rv_0,$$

provided that, for $r>1$, u_r, v_r are positive and $v_r>v_{r+1}$.

Using this result, it follows from (1) by taking

$$u_r=v_r=\frac{x^{2r+1}}{2^{2r+1}\underline{|2r+1}},$$

that $\sin^2\frac{x}{2}$ lies between W_n, W_{n+1}, where

$$w_r=\frac{x^{2r+2}}{2^{2r+2}}\Sigma\frac{1}{\underline{|2p+1}\,\underline{|2q+1}},$$

Σ denoting summation for $p+q=r$, $p=0, 1, 2, \dots r$, so that

$$w_r=\frac{x^{2r+2}}{2\,\underline{|2r+2}}.$$

Hence, using the formula $\cos x=1-2\sin^2\frac{x}{2}$ it follows from (1) that $\cos x$ lies between C_{n+1}, C_{n+2}, provided $-2\pi\leqslant x\leqslant+2\pi$.

E. J. NANSON.

III. ON CONVERGENCE OF SERIES.

HAVING established the expansion of functions without any reference to tests of convergence, we are able to enunciate the following simple test for the convergence of any series that may arise.

(We confine our attention to a series of which all the terms that we need to consider have the same sign.)

u_n being the n term of a series,

If $\dfrac{u_n}{u_{n-1}}$ can be expanded in (integral or fractional) powers of $1/n$, none of the terms after n^{-1} affect the convergence.

If $\dfrac{u_n}{u_{n-1}} \overline{\underline{<}} 1-\dfrac{a}{n}\ldots$ where a is greater than 1, the series is convergent.

If $\dfrac{u_n}{u_{n-1}} \overline{\underline{>}} 1-\dfrac{1}{n}\ldots$, the series is divergent.

1. The following theorem is well known: If Σu_i is a series, having $\dfrac{u_i}{u_{i-1}} \equiv \rho_i$, and Σv_i is another series having $\dfrac{v_i}{v_{i-1}} \equiv \sigma_i$, then if $\rho_i < \sigma_i$ for all values of i, and Σv is convergent, Σu is convergent; and if $\rho_i > \sigma_i$ and Σv is divergent, Σu is divergent.

For $$\sum_{n+1}^{N} u_i = u_n[\rho_{n+1} + \rho_{n+1}\rho_{n+2} + \ldots + \rho_{n+1}\rho_{n+2} \ldots \rho_N],$$

and $$\sum_{n+1}^{N} v_i = v_n[\sigma_{n+1} + \sigma_{n+1}\sigma_{n+2} + \ldots + \sigma_{n+1}\sigma_{n+2} \ldots \sigma_N].$$

If, therefore, every $\rho <$ the corresponding σ, $\Sigma u < \dfrac{u_n}{v_n}\Sigma v$;

$\therefore$ if Σv is finite when N is infinite, so is Σu.

That is, Σu is convergent if Σv is.

Similarly, if every $\rho >$ the corresponding σ, and Σv is divergent, so is Σu.

Thus, if we note the values of ρ for known series we can determine the convergence of unknown series.

2. If u_i is any 'derived' function $f'(i)$ which increases or diminishes with increasing i,

$$fi - f(i-1) < \left| \begin{matrix} f'i \\ f'(i-1) \end{matrix} \right|, \text{ and } f(i+1) - fi < \left| \begin{matrix} f'(i+1) \\ f'i \end{matrix} \right|;$$

$$\therefore f'i < \left| \begin{matrix} f(i+1) - fi \\ fi - f(i-1) \end{matrix} \right|; \quad \therefore \sum_{n+1}^{N} f'i < \left| \begin{matrix} f(N+1) - f(n+1) \\ fN - fn \end{matrix} \right|.$$

If, therefore, fN is infinite when N is indefinitely increased $\Sigma f'i$ is divergent, but if fN is finite, the series is convergent.

3. *Examples of known series.*

A. If $fx \equiv x^p$, then $f'x \equiv px^{p-1}$;

$$\therefore \sum_{n+1}^{N} i^{p-1} < \frac{1}{p} \left| \begin{matrix} (N+1)^p - (n+1)^p \\ N^p - n^p \end{matrix} \right|.$$

If p is negative, the series is convergent, because N^p is zero; and $\rho_n \equiv \left(1 - \frac{1}{n}\right)^{-p+1}$, which $> 1 - \frac{1-p}{n}$;

$\therefore$ any series for which $\rho_n \leqq 1 - \frac{1+\alpha}{n}$ where α is positive, is convergent.

B. If $p = 0$, $\frac{N^p}{p}$ becomes indeterminate; the form $\frac{1}{0}(N^0 - 1)$ suggests the logarithm; but the series in this form is better investigated afresh.

Thus if $fx \equiv \log x$, $f'x \equiv \frac{1}{x}$,

$$\text{for } \log x - \log(x-1) = -\log\left(1 - \frac{1}{x}\right) < \left| \begin{matrix} \frac{1}{x-1} \\ \frac{1}{x} \end{matrix} \right|;$$

$$\therefore \sum_{n+1}^{N} \frac{1}{i} < \left| \begin{matrix} \log(N+1) - \log(n+1) \\ \log N - \log n \end{matrix} \right|.$$

$$\therefore \Sigma \frac{1}{i} \text{ is (logarithmically) infinite.}$$

Also, $\rho_n = 1 - \frac{1}{n}$.

Hence any series for which $\rho_n \gtreqless 1 - \frac{1}{n}$ is divergent.

We have now to see for what values of α *infinitesimally* greater than 0, $1 - \frac{1+\alpha}{n}$ belongs to a convergent series, *e.g.* $\alpha = n^{-\beta}$ or $\alpha = (\log n)^{-\beta}$, where β is positive.

C. For this purpose it is natural to assume fN infinite in some smaller degree. Let fx be $(\log x)^p$.

Now $(\log x)^p - (\log y)^p < p(\log x - \log y) \left| \begin{matrix} (\log x)^{p-1} \\ (\log y)^{p-1} \end{matrix} \right|$

$$< p(x-y) \left| \begin{matrix} \frac{1}{y} \\ \frac{1}{x} \end{matrix} \right| \left| \begin{matrix} (\log x)^{p-1} \\ (\log y)^{p-1} \end{matrix} \right|.$$

Hence, if $(p-1)$ is negative,

$$(\log x)^p-(\log y)^p<p(x-y)\left|\begin{array}{l}\frac{1}{y}(\log y)^{p-1}\\ \frac{1}{x}(\log x)^{p-1}\end{array}\right|;$$

$$\therefore\ f'x \text{ is in this case } p\,.\,\frac{1}{x}(\log x)^{p-1};$$

$$\therefore\ \sum_{n+1}^{N}\frac{1}{i(\log i)^{p+1}}<\frac{1}{-p}\left|\begin{array}{l}(\log \overline{N+1})^{-p}-(\log \overline{n+1})^{-p}\\ (\log N)^{-p}-(\log n)^{-p}\end{array}\right|,$$

$\therefore$ the series is convergent if $p>0$, but divergent if p is a negative fraction.

Now, if $p>0$,

$$\rho_n=\frac{n-1}{n}\cdot\left(\frac{\log \overline{n-1}}{\log n}\right)^{p+1}, \text{ which} >\left(1-\frac{1}{n}\right)\left(1-\frac{1}{\overline{n-1}\log n}\right)^{p+1},$$

which $$>\left(1-\frac{1}{n}\right)\left(1-\frac{p+1}{\overline{n-1}\log n}\right), \text{ that is } 1-\frac{1+\frac{p+1}{\log n}}{n};$$

$\therefore$ any series for which $a\geqq\dfrac{p+1}{\log n}$ where p is positive, is convergent.

D. If $p=0$, $\dfrac{1}{p}\{(\log N)^p-(\log n)^p\}$ becomes indeterminate, but suggests the logarithm, and we find

$$\sum_{n+1}^{N}\frac{1}{i\log i}<\left|\begin{array}{l}\log\log(N+1)-\log\log(n+1)\\ \log\log N-\log\log n\end{array}\right|;$$

$$\therefore \text{ the series is divergent.}$$

Also $$\rho_n=\frac{n-1}{n}\cdot\frac{\log\overline{n-1}}{\log n},$$

which $$<\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\log n}\right)<1-\frac{1+\left(1-\frac{1}{n}\right)\frac{1}{\log n}}{n}.$$

$\therefore$ any series for which $a<\left(1-\dfrac{1}{n}\right)\dfrac{1}{\log n}$ is divergent.

Hence, if a is of the order $n^{-\beta}$ where β is positive, the series is divergent.

Hence the rule at the head of this article.

5. This test covers all ordinary cases; but the above investigation continued on similar lines by taking for fx

$$(\log\log x)^p, \ (\log\log\log x)^p, \text{ etc.},$$

leads us to the closer test:

If ρ_n expanded in descending order of magnitude is, or is greater than,

$$1-\frac{1}{n}-\frac{1}{n\log n}-\frac{1}{n\,.\log n\,.\log\log n}\cdots,$$

the series is divergent: if ρ_n is less than this it is convergent.

6. *Examples.* (From Chrystal, Part II., Exercise VIII., pp. 158, 159.)

Ex. 5. $u_n=\dfrac{1}{\sqrt{n^2-n}\{\sqrt{n}-\sqrt{n-1}\}}$,

$$\rho_n=\sqrt{\frac{n^2-3n+2}{n^2-n}}\cdot\frac{\sqrt{n-1}-\sqrt{n-2}}{\sqrt{n}-\sqrt{n-1}}$$

$$=\left(1-\frac{3}{n}+\frac{2}{n^2}\right)^{\frac{1}{2}}\left(1-\frac{1}{n}\right)^{-\frac{1}{2}}\left\{\left(1-\frac{1}{n}\right)^{\frac{1}{2}}-\left(1-\frac{2}{n}\right)^{\frac{1}{2}}\right\}\left\{1-\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\right\}^{-1}$$

$$=\left(1-\frac{3}{2n}\cdots\right)\left(1+\frac{1}{2n}\cdots\right)$$

$$\times\left\{1-\frac{1}{2n}-\frac{1}{8n^2}-1+\frac{1}{n}+\frac{1}{2n^2}\cdots\right\}\left\{\frac{1}{2n}+\frac{1}{8n^2}\right\}^{-1}$$

$$=\left(1-\frac{1}{n}\cdots\right)\left\{1+\frac{3}{4n}\cdots\right\}\left\{1-\frac{1}{4n}\cdots\right\}$$

$$=1-\frac{1}{2n}\cdots;\quad\therefore\text{ divergent.}$$

Ex. 12. $u_n=\dfrac{1}{n^a}\sum_1^n\dfrac{1}{i^a}$;

$$\therefore\ \rho_n=\left(\frac{n-1}{n}\right)^a\left\{1+\frac{1}{n^aS_{n-1}}\right\},\text{ where } S_{n-1}\text{ is written for }\sum_1^{n-1}\frac{1}{i^a}$$

$$=\left(1-\frac{a}{n}\cdots\right)\left(1+\frac{1}{n^aS_{n-1}}\right).\quad\text{This}>1-\frac{a}{n};$$

$$\therefore\text{ if } a \leqq 1,\text{ the series is divergent;}$$

but if $a>1$, we have in the article above,

$$\underset{n-1}{S}<1+\frac{1}{a-1}\left|\begin{matrix}2^{-a+1}-n^{-a+1}\\1^{-a+1}-(n-1)^{-a+1}\end{matrix}\right|;$$

$$\therefore\ n^a S_{n-1} < \text{const. } n^a - \frac{1}{a-1}\left|\begin{matrix} n \\ n\left(1-\frac{1}{n}\right)^{-a+1}\end{matrix}\right|;$$

$$\therefore\ \frac{1}{n^a S_{n-1}} = \frac{\text{const.}}{n^a} + \ldots, \text{ which does not affect the convergence.}$$

$$\therefore\ \rho_n = 1 - \frac{a}{n} \ldots, \text{ and the series is convergent.}$$

7. If $f(x_i) - f(x_{i-1}) < (x_i - x_{i-1}) \left|\begin{matrix} f'x_i \\ f'x_{i-1}\end{matrix}\right|$,

Then $$\sum_{n+1}^{N} (x_i - x_{i-1}) f'x_i > fx_N - fx_n,$$

and $$\sum_{n+1}^{N} (x_{i+1} - x_i) f'x_i < fx_{N+1} - fx_{n+1}.$$

Cauchy's Condensation Test is the particular case of this when $x_i \equiv k^i$ and $k > 1$.

W. N. ROSEVEARE.

REVIEWS.

Vorlesungen über Technische Mechanik. By Dr. A. FÖPPL. Vol. i. **Einführung in der Mechanik.** 3rd ed. pp. 424. 10 m. Vol. iii. **Festigkeitslehre.** 3rd ed. pp. 434. 12 m. (Teubner, 1905.)

In the first volume of this work Professor Föppl's object is to introduce a student who is familiar with the experimental facts of mechanics, to the mathematical treatment of the subject. A working knowledge of the *meaning* of such symbols as $\frac{ds}{dt}$ and $\int Pds$ is assumed, though little analytical skill is demanded of the reader.

The feature which distinguishes Professor Föppl's treatise from most English works on the same subject is the use of vectorial algebra and not merely vector addition. Both in these volumes and in his Introduction to Maxwell's Theory of Electricity, Professor Föppl constantly employs a vectorial notation and vectorial methods. He adopts the convention that, for scalar multiplication $i^2 = j^2 = k^2 = +1$.

Vector multiplication has at present received little attention from teachers of elementary mathematics in this country. This is probably due in part to the unfortunate controversy as to notation. [See Professor Henrici's communication to the British Association. *B.A. Report*, 1903, p. 55.]

But the use of almost any vector notation is of value as an aid to the memory and as directing attention to a physical quantity in itself instead of merely to its components along the axes.

An analogy is furnished by determinants. Who does not find it easier to recollect the result of eliminating x, y, and z from three equations (1), $a_1x + b_1y + c_1z = 0$, (2), and (3), in the form of a determinant?

To illustrate in a similar manner the use of vector multiplication; suppose a system of forces $R_1 \ldots R_n$ acting at points in space $r_1 \ldots r_n$.

The sum of the vector products $\Sigma[rR]$ gives the couple, the vector $\Sigma(R)$ the force acting at O, to which the system is reducible.

If then
$$R_l = X_l i + Y_l j + Z_l k, \qquad l = 1 \dots n,$$
$$r_l = x_l i + y_l j + z_l k,$$
since for vector multiplication
$$ij = k, \quad jk = i, \quad ki = j,$$
and
$$i^2 = j^2 = k^2 = 0,$$
we have $\Sigma[rR] = i\Sigma(Zy - Yz) + j\Sigma(Xz - Zx) + k\Sigma(Yx - Xy)$
and $\Sigma(Zy - Yz)$ is the component couple, in a right-handed system of axes, about the axis of x.

Again, suppose a system of axes moving about a fixed point with angular velocity $\Omega = i\theta_1 + j\theta_2 + k\theta_3$.

Suppose a vector quantity referred to the moving axes, as it would be by an observer unaware of the movement of the axes.

The point which is the geometrical representative so to speak of the vector quantity is supposed, that is to say, to be carried with the moving axes, and also to move relatively to them.

Now if $a = i\xi + j\eta + k\zeta$ is the vector and $q = \dfrac{d}{dt} + i\theta_1 + j\theta_2 + k\theta_3$ an operator, the vector product
$$\begin{aligned}[qa] = {} & i(\dot{\xi} - \eta\theta_3 + \zeta\theta_2) \\ & + j(\dot{\eta} - \zeta\theta_1 + \xi\theta_3) \\ & + k(\dot{\zeta} - \xi\theta_2 + \eta\theta_1)\end{aligned}$$
gives the rates of change of a, resolved into components along a right-handed system of axes fixed in space, and coinciding with the original position of the moving axes.

Putting the matter on the lowest ground these expressions appear to furnish a useful aid to the memory.

Professor Föppl's first volume contains many interesting applications—and he is always careful to discuss points of fundamental principle at adequate length.

The contents of the third volume on strength of materials and elasticity may be indicated by the titles of the chapters, which are as follows: General Investigation of Stress Components, Elastic Deformation, Bending of Straight Rods, Energy of Elastic Deformation, Curved Beams, Elastic Supports, "Built in" Plates, Resistance to Internal or External Pressure, Torsion, Fracture, Elements of the Mathematical Theory of Elasticity.

The mode of treatment adopted, making free use of the calculus, is more advanced than that of most English elementary text-books, though the book is not a mathematical treatise on the theory of elasticity.

To those desirous of going a little beyond an elementary discussion of Hooke's Law and bending moments the book should prove very interesting.

Reference may be made especially to the discussion of a beam subject to a non-axial thrust, and to the investigation by Castigliano's theorems of the 'continuous girder' or elastic beam resting on more than two supports. (See *Gazette*, vol. iii., p. 28.)

It may be mentioned that a French translation by M. Hahn, from the first edition of Professor Föppl's third volume, was published in 1901 by M. M. Gauthier-Villars. C. S. JACKSON.

Elementary Dynamics. By W. M. BAKER, M.A., Head Master of the Military and Civil Department of Cheltenham College. Second edition. (London: G. Bell and Sons, 1905. pp. 318. 4s. 6d.)

This is not an easy book to review. On the one hand praise is due to concise style and systematic treatment, and to the number and variety of the examples and their classification and arrangement. One is struck too by the way in which the difficulties of the subject are evaded.

But inasmuch as the subject does present certain difficulties and as the type of mind which never finds a difficulty in any subject has its disadvantages, ought this feature to be praised or blamed?

Mass is "quantity of matter" (p. 31); all motion is relative (p. 111), but force is measured by the product of mass and acceleration (p. 34) and yet is absolute (p. 35); variable velocity is measured "at any instant by the distance which would be passed over in unit of time if the point moved during that unit of time with the same velocity as at the instant under consideration." The rotatory definition last quoted occurs at p. 1, though it is true that velocity at a time t is obtained at p. 21 as the limit of $\frac{\Delta s}{\Delta t}$.

That the work is silent as to the history of dynamics and as to the practical applications of the subject must not be imputed to the author as a fault, for the book is written in view of certain examinations and it is not fair to blame an author who is subject to the conditions of a system for the consequences of that system. One is, however, surprised to read at page 123 "Angular velocity is measured in radians." C. S. JACKSON.

Oblique and Isometric Projection. By J. WATSON. (Edward Arnold. 3s. 6d.)

The principles of these projections are simply explained and their application to draughtsmen's work well illustrated. The diagrams are clear and of good size. We are afraid that mathematical teachers as a rule do not trouble themselves about such matters. This is to be regretted as they thus miss many a good exercise and illustration which they might use with effect when going through a school course of Theoretical Solid. A little of the 'fusing' discussed at one of the General Meetings would be good for both Practical and Theoretical work. E. M. LANGLEY.

An Introduction to Projective Geometry and its Applications, an Analytic and Synthetic Treatment. By ARNOLD EMCH, Ph.D. (New York: Wiley & Sons. London: Chapman & Hall. 1905. pp. 267.)

This seems to us a singularly ineffective book. It must prove disappointing to a reader who wishes to gain from it some definite ideas of projective geometry; and it only differs from any ordinary book on geometry in being more than usually meagre and inaccurate. Projective geometry is the geometry which is founded on projection, and, as generally understood, makes no metrical assumptions, but regards all figures as being immovably fixed in space. It is the most fundamental of all geometries, being based on the fewest axioms. Even if metrical ideas were permitted we should still regard projective geometry in its beginnings as pure geometry. Dr. Emch, however, is of a different opinion; he makes no enquiry as to the foundations and limitations of the science, and defines projective ranges quite apart from projection. His work, as he says, has a utilitarian purpose, although this is not very apparent in the greater part of the book. But, being utilitarian, why should the terms descriptive and graphical be rejected, and a term be chosen which is specially used in a sense opposed to these?

The book contains five chapters devoted respectively to General Considerations, Collineation, Conics, Cubics, and Mechanics. Some of the subsections are involution, perspective, central projection, orthographic projection, polar involution, properties of the centre, diameters, asymptotes, and foci of conics, and the theorems of Pascal and Brianchon. The last chapter includes the geometry of stresses, and a description of a number of linkages.

We can only refer to some of the weak points in the exposition. The questions of the independence and consistency of a given system of linear equations are generally ignored. On p. 132 it is indeed shown that a set of six linear equations are not equivalent to more than five; but it is there stated that they have a unique solution for five unknowns, although the six equations have been formed from four only. The theorem professedly proved is of course false. On p. 23 the enumeration of independent constants is incorrect. The description on pp. 27, 28 of the complete quadrilateral is actually that of the complete quadrangle; and the remarkable theorem is given:—"In every complete quadrilateral a pair of sides always forms a harmonic pencil with the two concurrent diagonals." It would be difficult to cram more mistakes into a statement of equal brevity. Six lines lower down the terms quadrilateral and diagonal are used in the proper sense, *i.e.*, in a sense quite contrary to that in which they have been defined. The terms quadrilateral, quadrangle, and quadruple are elsewhere used indiscriminately. As regards the polar involution of the circle we read (p. 43):—"For every ray through P a pair of poles and a pair of polars are obtained which are harmonic to A and B, and to a and b, respectively. In this manner an *involution of coincident poles and polars* arises." The italics are those of the book. This is the sole explanation of the meaning of polar involution. We fear it will leave the reader, as it leaves us, unenlightened. According to the explanation given on p. 68, the Principle of Duality means nothing more than that a curve may be regarded as a locus of points or as an envelope of lines. Besides the use of quadrilateral for quadrangle, coincident is used for incident (p. 127), and bisection for harmonic section (pp. 89, 142). Definitions are as a rule omitted; and the whole treatment is entirely informal and vague.

In the preface it is stated that Enriques "in his book," *Projektive Geometrie*, "lets the fundamental elements of the first order be generated by motion!" This is misleading, if, as we presume, forms are meant by elements. Enriques only uses motion by way of illustration. His geometry is founded on carefully formulated axioms, in which the idea of motion has no part. F. S. M.

Geometrical Conics. By CAUNT AND JESSOP. (London: Edward Arnold. Price 2s. 6d.)

This is a small book of 80 pages. The authors define the conic as the section of a right circular cone, and from this definition they deduce the focus and directrix property, the bifocal property and the construction for describing the curve by mechanical methods. A short account is given of the polar properties of the circle, on which, by the method of projection, the authors base the theory of pole and polar and of conjugate diameters in the conic. The book deserves commendation for giving prominence to the idea of the continuity of the different species of the conic. We think, however, that the explanation of the method of projection is imperfect. In Chapter I. the conic is defined as the section of a *right* circular cone, but in Chapter III. the section of *any* cone on a circular base is called a conic; and it is not proved that every conic, defined by the focus and directrix property, can be projected into a circle. R. H. PINKERTON.

The First Book of Euclid's Elements, with a Commentary based principally on that of Proclus Diadochus. By W. B. FRANKLAND. Pp. xvi, 139. 1905. (Cam. Univ. Press.)

Mr. Frankland has followed up his excellent *Story of Euclid* by an "anthology of the best commentary on any part of the Elements ever composed." We observe with regret from the preface that untoward circumstances of a private nature have prevented the earlier appearance of the book before us. But such is its quality and that of its predecessor that it is evident that Mr. Frankland possesses a large measure of the equipment required in the writer on such a topic as the parallel postulate. We therefore welcome the hint given us in the first chapter, that some day there will appear from the skilful and learned pen of the author an historical and critical study of that problem. We could wish we had space for many quotations from this witty and charming volume. Here is one which would have gladdened the heart of the author of *Euclid and his Modern Rivals*, and of *Alice in Wonderland*: "Easy introductions to Euclidean dogmatics do not expose clearly and precisely the principles at their bases—they need to be exposed. Such plebeian roads . . . were beginning to appear, like much else reckoned modern, in the *Commentary of Proclus*." "Parallelism is similarity of position. This is good illustration, but bad analysis." We must content ourselves with saying that these "innocent studies" form an eminently valuable addition to the history of Geometry, and we may add, an attractive volume.

Euclid's Parallel Postulate. Its Nature, Validity, and Place in Geometrical Systems. By J. W. WITHERS, Ph.D. Pp. vii, 192. 1905. (Kegan, Paul.)

This book is the thesis presented to the Philosophical Faculty of Yale University for the degree of Ph.D. by the author. He gives an account of the history of the postulate up to the time of the discovery and development of non-Euclidean systems, and dwells upon the significance of the latter. He then embarks on the philosophical side of his theme. First he discusses the psychology of the parallel postulate and its kindred conceptions, showing that its validity is not to be settled by any empirical investigations. Next he deals with the number and variety of possible space geometries. Finally he shows the inferences that follow as to the nature of space when the validity of the parallel postulate is denied. On the whole we may consider the book as a compilation in which the problem and its history are presented accurately and attractively. The author seems to have read everything (in English) that ought to have been read for the purpose, but we do not rise from the perusal of his book with the feeling that anything has been added to what has been said many times before. The book, however, has a certain value, but we are sorry to add that its value is considerably impaired by a large number of errors. Thus we have—"Eudemas" for "Eudemus" (p. 2) and "Germinus" for "Geminus" (p. 5). On p. 6 we are told that the most critical efforts of modern times have not destroyed, but have strengthened Euclid's claims to rigour, an opinion which would find no support from Mr. Bertrand Russell. On p. 7 "Autolycus" becomes "Antolycus," and Sir Henry Savile receives an extra "l" in his surname. The meaning of the Latin quotation is destroyed by "elucendis" for "eluendis" and by an omission. Playfair's axiom (p. 11) is credited to "Ludlum" instead of to "Ludlam." On p. 13 we have La Place, and the writer speaks of T. Perronet Thompson, "of Cambridge," brilliantly stating the insufficiency of twenty-one different attempts to dispose of the parallel postulate. The number should be 30 and not 21, and Thompson could hardly be described as "of Cambridge." It is true that he was at Queen's at the mature age of fifteen, but he went into the Navy at twenty, and is not likely to have done much that was brilliant within that period. After serving in the Navy for three years he exchanged into the Army, and eventually attained the rank of General. His connection with Cambridge can have been but slight, for his work as joint-editor of the *Westminster Review*, and the claims of Parliament (he was M.P. for Hull and for Bradford) must have kept him for the most part in London. On p. 14 we have "geomatriae" and "Legendri." The *Encyclopaedia* reference on p. 19 should be p. 666. On p. 38 (η, θ) should be (r, θ). On p. 43 "ejusdem" is wrongly spelled, and on the next page we have "spazzii" for "spazii" and "constante" for "costante." "Motion" for "notion" on p. 143 spoils the sense of the sentence. In the Bibliography at the end we think we are correct in stating that not a single French accent is rightly placed. Under the name Beltrami we have "spazii" rightly spelled this time, but "constanta" again, and "interpretazioni" for "interpretazione." Under *Bolyai* there is "subliminis" for "sublimioris" and "evidentique" for "evidentiaque." Under *Bonola* we have "Matematische" for "Matematiche" and "Fondamenta" for "Fondamenti." Under *Calinon* there is "Matematique" for "Mathématiques." The firm of Macmillan is throughout given as MacMillan. *Couturat* is omitted from the list. The reference to *Mind* under the title *Helmholtz* should be Vol. II. and not III. Under *Klein* we have "Reimann,' and under *Pasch* "neure" for "neue," "Euclidis" for "Euclides" and "philosphe" for "philosophie"—but why extend the tale? The author either knows nothing of the languages from which he quotes or has been amazingly indifferent to the necessity of correcting his proofs.

Easy Mathematics, chiefly Arithmetic; being a collection of Hints to Teachers, Parents, Self-taught Students, and Adults, and containing Most Things in Elementary Mathematics Useful to be Known. By Sir Oliver Lodge. Pp. xv, 436. 1905. (Macmillan.)

There is a quaint eighteenth-century flavour about the sub-title of an interesting volume, which owes its existence to the author's conviction that mathematics may be taught very much better than is the case in most of our schools. Accordingly he has tried to make the subject lively and interesting. "Dulness and bad teaching are synonymous terms." Teaching that is not characterised by these qualities is actually harmful. "A few children are born mentally deficient, but a number are gradually made so by the efforts made to train their growing faculties." These be bitter words, and we devoutly hope the indictment is undeserved, at any rate with regard to a large number of our secondary schools. Sir Oliver's wide experience and the prominent position he has won for himself in the world of science, together with the opportunities for observation he enjoys in the important position he now holds, entitle his views to be carefully studied and with respectful attention, and most teachers will read with pleasure a book written in such a breezy style, thoroughly sane from cover to cover; while few will find no hint in its pages by which their attitude towards the unformed mind as teachers of the subject is not capable of improvement. Some of the points made are quite excellent. Among the examples on the unitary method are some which most certainly will find out for a teacher if his boys are applying their method in a thinking way. "If two peacocks can awaken one man, how many can awaken six? If a camel can stand a load of 5 cwt. for six hours, how long can he stand a load of ten tons?"

Cours de Géométrie Analytique de deux Dimensions. Des Sections Coniques. By H. MANDART. Pp. 574. 1905. (Wesmael, Namur.)

We are glad to draw attention to this volume on Analytical Conics for two reasons. In the first place we do not come into close enough contact with the really excellent work which is being done in the Belgian secondary schools, technical schools, and colleges, and we get little opportunity of seeing with what apparatus of text-books are obtained results which are making Belgium a force to be reckoned with in the industrial and scientifical world. And in the next place for the sake of the book itself, inasmuch as it initiates a departure from the usual traditionary treatment of the subject. The books on Conics for the Belgian secondary schools which we have seen have seemed to differ in some respects from the more advanced works intended for the consumption of children of a larger growth. In the former case there is no attempt to impart general views; the treatment differs but little from that to which we are accustomed, and the geometrical point of view reigns supreme. The conics are discussed each in turn; rarely does the general equation appear. But directly the student makes his way to the Universities he has to begin the subject over again. He meets with the ternary quadratic form; projective properties are utilised for the relations between the partial derivatives of the form, and the metrical properties are treated with the aid of the cyclic points. For some time past M. Mandart has been dissatisfied with this way of dealing with the elements of the theory of curves of the second order, and the volume under notice is the outward and visible sign of the methods of teaching which he has been trying for some years past. Let us say in a nutshell that the whole aim and object of the book is to keep before the student the rôle of invariants, while at the same time using nothing but the elements of invariantology. Whether the method will prove satisfactory from the teaching point of view or not we will not attempt to hazard a conjecture. It might be tested, first with picked boys, and then with the rank and file if the results seem promising. The book forms a complete treatise. The first two parts deal with the point, line, and circle, and with lines of the second degree. The third takes up the general theory of conics, and the fourth is devoted to homogeneous trilinear coordinates. General principles are always kept in sight, and nothing is slurred or passed over without close examination. Plenty of examples are worked, and plenty set for the student as exercises. The whole subject is treated in a manner which reveals the hand of a teacher with a mastery of his work, so complete and homogeneous is the book, and so lucid is the exposition. We would like to hear if any teacher finds it possible to make even a partial attempt to teach Analytical Conics on these lines, feeling sure that the experiment is worth a trial.

Mathematical Recreations and Essays. By W. W. ROUSE BALL. Pp. 388. 4to Edition. 1905.

It is about three hundred years since Claude Gaspar Bachet, Sieur de Meziriac, wrote in his dedication to Monsieur le Comte de Tournon of his *Problèmes Plaisants et Delectables*: "Je vous offre des jeux, mais qui sont, à mon avis, dignes de votre bel esprit, et capables de lui fournir quelquefois un agréable divertissement." One hundred years later Leibnitz, in a letter to Montfort, expressed the wish "qu'on eût un cours entier des jeux, traités mathématiquement." In 1694 appeared Ozanam's *Recréations Mathématiques et Physiques*, which was a compilation from various sources, Ozanam's own share being of but indifferent value. Still, his book was a success and went through many editions, the last being in one volume in 1840. In 1892 Mr. W. W. Rouse Ball published a volume entitled *Mathematical Recreations and Problems of Past and Present Times* (Messrs. Macmillan). A fourth edition has now made its appearance with the 240 pages of the original increased to almost 400. There are three new chapters, dealing respectively with the History of the Mathematical Tripos at Cambridge, Mersenne's Numbers, Cryptography and Ciphers. Part II., which was headed "Mathematical Problems and Speculations," is now altered to "Miscellaneous Essays and Problems." The problem of Mersenne's numbers is to find the values of p which make $2^p - 1$ a prime—*i.e.* it is a particular case of finding the factors of $a^n - 1$. The peculiar interest attaching to Mersenne's researches is that to this day no one can say how he or his contemporaries arrived at their results. Copious references in this chapter direct the student where to look for information on the various cases, and it is with mournful resignation that we learn that no machines have yet been devised for the detection of primes among numbers so large as those given by Mersenne. The chapter on Cryptography and Ciphers is full of information. Here we have the historical ciphers of Julius Caesar, Augustus, Bacon, Charles I., Pepys, and—sinister conjunction of names—those of De Rohan and Marie Antoinette, though the De Rohan is not the great Cardinal, but one of humbler rank—a mere Chevalier, who probably lost his life through being unable to read the cipher message sent him by his friends. As there may be occasions in the lives of the least adventurous among us when it may be useful and advisable to baffle the curiosity of an occasional village postmistress, there will be times when this chapter will be eagerly read.

The old chapter on the "Constitution of Matter" has been reconstructed, as, indeed, was inevitable in the light of recent researches. The sections added deal with the hypothesis of an elastic solid aether and Sir William Thomson's labile aether. The dynamical theories now treated are the hypothesis of the vortex ring, of the vortex sponge, of the aether squirt, of the electron, and of the bubble. There is also a short account of the latest speculations due to investigations on radio-activity. Some account might have been given of Le Bon's speculations, a convenient summary of whose suggestive book has been given recently in the *Athenæum*. Roughly speaking, his theory may be summed up as follows: All matter is slowly dissociating; electricity is the intermediate state through which matter passes on its passage to the aether from whence it came. But to return to Mr. Rouse Ball. We might naturally expect from the author of *The History of the Study of Mathematics at Cambridge* a lucid and systematic account of the Tripos, and we are not disappointed. We see how the word has changed its meaning as few words have, "from a thing of wood to a man, from a man to a speech, from a speech to sets of verses, from verses to a sheet of coarse foolscap paper, from a paper to a list of names, and from a list of names to a system of examination." With the author, we regret to note the disappearance during the last ten years of the famous Tripos verses, which in their time caused a good deal of harmless and innocent amusement. "Mr. Tripos" had been an unchartered libertine for three hundred and twenty years, off and on; he was allowed to say anything he liked in these verses "so long as it was not dull and was scandalous"; and, to quote the exhortation of the University officials, he always remembered of recent years while exercising his privilege of humour to be modest withal. As well might we call for the abolition of the Westminster Play! We must add that a considerable amount of new matter has been added in this edition, much of it of an elementary character and such as to be easily understanded of the vulgar.

W. J. G.

On the Traversing of Geometrical Figures. By J. Cook Wilson. 1905. (Clarendon Press.)

To men who read mathematics occasionally for amusement, but whose serious work has usually lain in other directions, this book will almost certainly be of interest. It needs no close or concentrated attention, and deals in a simple and convincing manner with the discussion of a mathematical puzzle which may easily have already occurred to the reader as one deserving of solution. Under what circumstances is it possible to trace out a given geometrical figure without removing the pen from the paper? For the solution arrived at, which is very simple, and kindred topics, the reader must be referred to the book itself. Part III. of the treatise is devoted to the application of the Principle of Duality, and the "Addendum," published subsequently to the treatise, deals with the same matters. Here the author is more difficult to follow, and indications are not wanting that he is not himself one of the elect to whom the inner mysteries of modern Pure Mathematics have been laid bare.

W. H. Young.

MATHEMATICAL NOTES.

177. [**K. 23. c.**] *Projection of Diagrams.*

In an article on "Projection of Mathematical Diagrams" in the *Penny Cyclopædia*, De Morgan remarks:

"The diagrams by which mathematical students (and even writers) represent their solid figures are generally so imperfect that it may be worth while to explain how in all cases of sufficient importance a good drawing may be made with very little trouble. . . ."

In this Note De Morgan is followed closely.

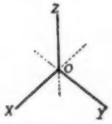


FIG. 1.

The projection is supposed to be ORTHOGRAPHIC, in which the eye is at an infinite distance, and all parallels project into parallels, etc.

Let OX, OY, OZ (Fig. 1) be the intended projection of the three axes of coordinates $O'X'$, $O'Y'$, $O'Z'$, the dark lines being supposed to belong to that quarter of space in which lies a line drawn from the eye to the origin. Each of the angles YOZ, ZOX, XOY is then greater than a right angle.

Let these angles be denoted by X, Y, Z, and let the projections of a line of given length l in the directions of $O'X'$, $O'Y'$, $O'Z'$, be x, y, z, then

$$x : y : z = \sqrt{-\sin 2X} : \sqrt{-\sin 2Y} : \sqrt{-\sin 2Z}.$$

For the proof De Morgan refers to the *Cambridge Mathematical Journal*, No. 8, p. 92.

But we may proceed as follows:

Let A, B, C be the points in which $O'X'$, $O'Y'$, $O'Z'$ meet the plane of projection. Draw $O'D$, $O'E$, $O'F$ perp. to BC, CA, AB and join AD, BE, CF.

$$AB^2 = AO'^2 + O'B^2 = AO'^2 + O'D^2 + BD^2 = AD^2 + DB^2,$$

$$\therefore\ AD \text{ is perp. to } BC;$$

$$\therefore\ O \text{ lies on } AD.$$

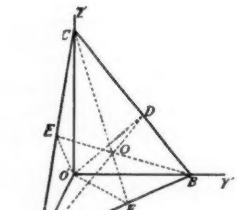


FIG. 2.

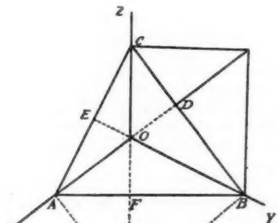


FIG. 3.

Similarly O lies on BE and CF, *i.e.* O is the orthocentre of ABC.

$$\therefore\ \angle BOC = 180° - A,$$

$$\text{and } \therefore\ AO'^2 = AE \,.\, AC = AO \,.\, AD.$$

(In Fig. 3 the triangle $AO'B$ is folded into the plane ABC, which is taken as that of the paper.)

Then
$$\frac{AO^2}{AO'^2}=\frac{AO^2}{AF.AB}=\frac{AO^2}{AO.AD}=\frac{AO}{AD}=\frac{AO.BC}{2\Delta}$$
$$=\frac{2R\cos A.2R\sin A}{2\Delta}$$
$$=\frac{R\sin 2A}{\Delta}$$
$$=-\frac{R}{\Delta}\sin 2X;$$
$$\therefore\ x=l\sqrt{-\frac{R}{\Delta}\sin 2X}.$$

Similarly for y and z.

For convenience take $l\sqrt{\frac{R}{\Delta}}$ as unity, then the following table gives the length of the projections for the angles entered.

91°	·187	106°	·728	121°	·940
92	·264	107	·748	122	·948
93	·323	108	·767	123	·956
94	·373	109	·785	124	·963
95	·417	110	·802	125	·969
96	·456	111	·818	126	·975
97	·492	112	·834	127	·980
98	·525	113	·848	128	·985
99	·556	114	·862	129	·989
100	·585	115	·875	130	·992
101	·612	116	·888	131	·995
102	·638	117	·900	132	·997
103	·662	118	·911	133	·999
104	·685	119	·921	134	1·000
105	·707	120	·931	135	1·000

Since
$$\sin 2(270^\circ-A)=\sin(540^\circ-2A)=\sin 2A,$$
the length corresponding to an angle greater than 135° may be found by taking the reading for its defect from 270°; thus the length corresponding to 160° is ·802, the same as for 110°.

The use of the table is as follows:

Suppose the angles X, Y, Z to be 123°, 105°, 132°, as in Fig. 1. Opposite the angles put down the numbers belonging to them in the table, and opposite to each number the corresponding coordinate

X	123°	·956	x
Y	105°	·707	y
Z	132°	·997	z

These numbers show the proportions which the projections of equal lines bear to one another on the three axes. Thus

$$\frac{\text{Projection of 1 ft.} \parallel \text{to } O'X'}{\text{Projection of 1 ft.} \parallel \text{to } O'Y'}=\frac{\cdot 956}{\cdot 707}.$$

The isometrical perspective of Prof. Farish is the simplest case of this, namely, that in which the angles are each 120°.

See Sopwith's "Isometrical Drawing" for an elaborate account of this and its various applications.

E. M. LANGLEY.

178. [A. 1. c.] *Approximation to the r^{th} root of a number.*

In a review published in the May number of this *Gazette*, Mr. C. S. Jackson has suggested methods of arriving at a certain approximation to the r^{th} root of a number N. The following method of dealing with this problem is very elementary, in that it only assumes the Binomial Theorem *for a positive integral index* and it is closely allied to the ordinary arithmetical method of extracting roots.

Let a be the first approximation to the root, and let

$$\sqrt[r]{N}=a+x.$$

$$\therefore\ N=a^r+r\,.\,a^{r-1}x+\frac{r(r-1)}{2}a^{r-2}x^2+\dots,$$

$$i.e.\ \ x=a\,.\,\frac{N-a^r}{r\,.\,a^r}\Big/\left(1+\frac{r-1}{2}\,\frac{x}{a}+\frac{(r-1)(r-2)}{2\,.\,3}\,\frac{x^2}{a^2}+\dots\right).$$

The first approximation for x is 0.

The second is $x_2=a\,.\,\dfrac{N-a^r}{r\,.\,a^r}$. This is the approximation which is generally used in seeking the next digit in arithmetical work.

The third approximation is

$$x_3=x_2\Big/\left(1+\frac{r-1}{2}\,\frac{x_2}{a}\right)=\frac{2a(N-a^r)}{(r+1)a^r+(r-1)N}.$$

The corresponding approximation to $\sqrt[r]{N}$ is $a+x_3$, which is

$$a\,.\,\frac{(r-1)a^r+(r+1)N}{(r+1)a^r+(r-1)N}.$$

This solution can therefore be obtained without any preliminary assumption as to its form.

It is of some interest to notice that this work can be set out in the way which is generally used for square and cube root, thus:

$$\left[\text{N.B.}—x_2\equiv\frac{N-a^r}{ra^{r-1}}\right]$$

$$\begin{array}{r|l} N & a+x_2-\dfrac{(r-1)\,.\,x_2^2}{2a}\dots \\ a^r & \\ \hline r\,.\,a^{r-1}+\dfrac{r(r-1)}{2}a^{r-2}x_2+\dots & (N-a^r) \\ & (N-a^r)+r(r-1)x_2^2a^{r-2}/2+\dots \\ \hline r\,.\,a^{r-1}+\dots\dots\dots\dots & -r(r-1)x_2^2a^{r-2}/2-\dots \\ & -r(r-1)x_2^2a^{r-2}/2-\dots. \\ \hline \end{array}$$

When the higher powers are neglected this is equivalent to

$$\sqrt[r]{N}\doteqdot a+x_2-\frac{r-1}{2}\,\frac{x_2^2}{a}$$

$$\doteqdot a+x_2\Big/\left(1+\frac{r-1}{2}\,\frac{x_2}{a}\right),$$

the same result as before. F. J. W. Whipple.

179. [A. 3. k.] *Cubic Equations.*

Attention may be called to a Note by Mr. D. Biddle on a direct method of solving (to any required degree of approximation) cubic equations when irreducible by Cardan's method. It is to be found in the *Educational Times* for July 1st, 1905. The process is "neither laborious nor prolonged, and though expedited by the use of logarithms and reciprocals, is in reality independent of all tables."

180. [K. 2. a.] *Note on Simson's Line.*

A, B, C, D are points on a circle, centre O, radius R; K, k orthocentres of the triangles ABC, DBC.

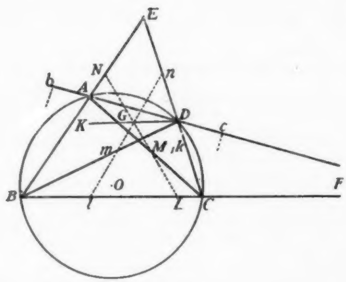
Then AK, Dk are parallel and each equal to twice the perpendicular from O on BC, hence $AKkD$ is a parallelogram, and its diagonals bisect each other in G.

The rectangular hyperbola through A, B, C, D must also pass through K and k; so that G is its centre, and the nine-point circles of the triangles ABC, DBC, ABD, ACD intersect in G.

If LMN, lmn be the Simson lines of D and A with respect to the triangles ABC, DBC respectively, we may assume as known that LMN bisects DK, *i.e.* it passes through G.

Similarly the pedal lines of A, B, C with respect to the triangles DBC, ACD, ABD also pass through G.

Let BA, CD meet in E; and BC, AD meet in F.



Now $DNA = AnD = 90°$, $\therefore$ $ANnD$ is cyclic, and Nn, AD are antiparallel with respect to EAB, EDC. Hence Nn is parallel to BC, and in the same way Mm is parallel to BC.

The perpendicular from G on BC bisects Ll; $\therefore$ $GL = Gl$, $GM = Gm$, and $GN = Gn$.

Thus the four Simson lines not only co-intersect in G, but are all of the same length, and have their segments measured from G to the sides of the respective triangles all equal to one another.

Let Bb, Cc be perpendiculars to AD; then

$$Gb = Gc = Gl = GL.$$

Thus G is the circum-centre of the triangle lbc, which is similar to ABC (for $\angle lbc = \angle lBA$ and $\angle lcb = \angle BCA$).

Hence G is the centroid of masses $\sin 2A$, $\sin 2B$, $\sin 2C$ at l, b, c respectively.

But $$\angle Abm = \angle ABD = \angle ACD = \angle Acn.$$

$$\therefore\ bm \text{ is parallel to } cn.$$

Thus G is also the centroid of masses $\sin 2A$, $\sin 2B$, $\sin 2C$ at l, m, n respectively, or (as the respective distances from G are equal) of those masses at L, M, N. Similar results hold for the other similar lines.

Lastly, let us take the proposition for which Mr. R. F. Davis (*Math. Gazette*, No. 48, p. 116) desires a geometrical meaning.

Suppose A, B, C fixed and D variable.

$$\Sigma OL^2 \sin 2A = OG^2 \Sigma \sin 2A + \Sigma GL^2 \sin 2A$$
$$= OG^2 \Sigma \sin 2A + \Sigma Gl^2 \sin 2A$$
$$= \Sigma Ol^2 \sin 2A$$
$$= R^2 \Sigma \sin 2A - Bl \, . \, lC \sin 2A - Bm \, . \, mD \sin 2B$$
$$+ Cn \, . \, nD \sin 2C.$$

But $Bl \, . \, lC \sin 2A = 2AB \, . \cos B \, . \, AC \, . \cos C \sin A \cos A = 8k^2 \, . \, \Pi \sin A \, . \, \Pi \cos A,$

$$Bm \, . \, mD \sin 2B = 2AB \cos ABD \, . \, AD \cos ADB \, . \sin B \cos B$$
$$= 2AC \, . \cos ACD \, . \, AD \cos ADn \, . \sin C \, . \cos C$$
$$= Cn \, . \, nD \sin 2C.$$

Thus $$\Sigma OL^2 \sin 2A = 4R^2 \Pi \sin A (1 - 2\Pi \cos A).$$ E. P. ROUSE.

181. [L^1. 2. b.] *A solution, not by elliptic functions, is wanted of the following:*

Given 5 lines *abcde* in a plane, it is known that the pairs of points

ab, *ce* ; *bc*, *da* ; *cd*, *eb* ; *de*, *ac* ; *ea*, *bd*

are in a collineation. Prove that the fixed triangle of this collineation is self-polar as to both the conic on *ab*, *bc*, *cd*, *de*, *ea* and the conic on *ac*, *ce*, *eb*, *bd*, *da*. J. MORLEY.

182. [K.] *Proofs of Euler's Theorem, etc.*

Lemma :—The resultant of successive reflexions in two planes intersecting at an angle α and in a line l is a rotation through 2α about l.

To find the resultant of a rotation through α about an axis OA followed by a rotation through β about an axis OB.

Take OC such that the angle between the planes OAB and $OAC = \frac{1}{2}\alpha$ and the angle between OBC, $OBA = \frac{1}{2}\beta$. Then the required resultant $\equiv$ the resultant of successive reflexions in OAC, OAB, OBA, $OBC \equiv$ the resultant of successive reflexions in OAC, $OBC \equiv$ a rotation about OC through twice the angle between OCB, OCA.

To reduce any displacement of a rigid body to its simplest form.

(1) Let A, B, ... be the initial and α, β, ... the final positions of a set of parallel planes. Then it is readily shown that A and α, B and β, ... meet on a fixed plane p. Let A', B', ... and α', β', ... be the initial and final positions of a second set of parallel planes, and let p' be the locus of the intersections of A' and α', B' and β', ... Let S be any point on the intersection l of p and p' and let M, μ, M', μ' be the planes through S parallel to A, α, and A', α'. Let l and S be moved by the displacement to λ and σ. Since S lies on M and M', σ lies on μ and μ'. Hence every point of l is moved parallel to the intersection of α and α', and therefore l and S are moved to λ and σ either (i) by a translation, or (ii) by a rotation through some point O on l. In either case the displacement of the body is completed by a rotation about λ. Hence the total displacement is equivalent (i) to a translation followed by a rotation, *i.e.* in general to a screw ; (ii) to two rotations about lines through O, *i.e.* to a rotation.

(2) Let two points of the body whose final positions are A' and B' be brought to A and B by a translation which brings any one point to its final position O. Let β be the reflexion of B in the plane p bisecting AA' at right angles, and let the plane q bisect $B'\beta$ at right angles. Then p passes through O since $OA = OA'$, and q passes through O and A' since $O\beta = OB = OB'$ and $A'\beta = AB = A'B'$. Now the points O, A, B are brought to O, A', B' by successive reflexions in p and q, *i.e.* by a rotation about the intersection of p and q. But three points determine the position of a body, and therefore the displacement is equivalent to a translation followed by a rotation.

These proofs can be modified to suit the case in which the initial and final positions of a figure are enantiomorphous. HAROLD HILTON.

NOTICE.

On the occasion of the Fourth International Congress of Mathematicians which will be held in Rome in 1908, an international prize for Geometry will be awarded by the *Circolo Matematico di Palermo.* In honour of its founder, Professor Guccia, it will be called the Guccia medal, and will consist of a small gold medal and the sum of 3000 francs. Since the Steiner prize of 1882 was awarded the theory of gauche algebraical curves has not received the attention it deserves. There is in fact quite a number of important problems connected with algebraical curves, such as their classification, the study of canonical curves of given deficiency, and the like, which have yet to be solved. And but few theorems in which there is a limitation either to a real field or to a given domain of rationality are known with respect to algebraical gauche curves. The Guccia medal will be awarded to the author of the most important memoir making an advance in the theory of these curves. No other conditions are prescribed either as to the methods of research or as to the problems to the investigated. Should nothing be presented worthy of note, the prize may be awarded to the author of a memoir making an important advance in the theory of algebraical surfaces or of other algebraical manifoldnesses.

The memoirs must be in Italian, French, German or English, and, with the exception of the formulae, must be typewritten. They must not have been previously published. Three copies inscribed with a motto must reach the President of the *Circolo Matematico di Palermo* before July 1st, 1907, accompanied by a sealed envelope containing the motto, name, and address of the author. The prize winner will receive 200 free copies of the memoir, which is to be printed in one of the publications of the *Circolo.* If the prize is not awarded for a new memoir, it may be given to the author of a published work on the theories mentioned, provided that it appears between the date of this notice and July 1st. 1907.

The "judges" are Professors Noether, Poincaré, and Segre.

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